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New exactly solvable isospectral partners for $\mathcal{PT}$ symmetric potentials

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Abstract

We examine, in detail, the possibility of applying the Darboux transformation to non-Hermitian Hamiltonians. In particular we propose a simple method of constructing exactly solvable $\mathcal{PT}$ symmetric potentials by applying Darboux transformation to higher states of an exactly solvable $\mathcal{PT}$ symmetric potential. It is shown that the resulting Hamiltonian and the original one are pseudo supersymmetric partners. We also discuss application of the Darboux transformation to Hamiltonians with spontaneously broken $\mathcal{PT}$ symmetry.

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1. Introduction

Ever since it was conjectured by Bender *et al* that some non-Hermitian Hamiltonians exhibiting symmetry under the combined transformation of *parity* ($\mathcal{P} : x \rightarrow -x$), and *time reversal* ($\mathcal{T} : i \rightarrow -i$) admit real eigenvalues [1, 2], non-Hermitian Hamiltonians have been the basis of many recent works on $\mathcal{PT}$ symmetry and pseudo-Hermiticity [3, 4], because of intrinsic interest and their possible applications in molecular physics, quantum chemistry, superconductivity, quantum field theory and others.

On the other hand, there are not many examples of exactly solvable complex potentials (both $\mathcal{PT}$ invariant as well as otherwise). However, as in the Hermitian case, there have been attempts to expand the class of exactly solvable non-Hermitian potentials by using different methods [5–7]. In the Hermitian case a popular method for obtaining new exactly solvable potentials is to apply the Darboux transformation [8] to the ground state of an exactly solvable potential. However, when applied to the excited states, this transformation produces not just one isospectral potential, but a number (depending on the nodes of the wavefunction) of nearly isospectral potentials which are defined not over the whole domain, but in disjoint intervals [9]. Here our objective is to apply the Darboux transformation to non-Hermitian potentials and it will be shown that for such potentials, it is possible to have wavefunctions without nodes on the real line, by a reasonable choice of parameters. In the present paper, we shall use this result to construct non-trivial isospectral partners of exactly solvable complex potentials. In particular, we shall apply the Darboux transformation to the well-known $\mathcal{PT}$ symmetric Scarf II potential

$$V(x) = -V_1 \operatorname{sech}^2 x - \mathrm{i}V_2 \operatorname{sech} x \tanh x \qquad V_1 > 0 \quad V_2 \neq 0 \tag{1}$$

and generate a series of new exactly solvable non-Hermitian potentials with real spectrum.

We note that in the case of Hermitian quantum mechanics, the Darboux transformation is equivalent to supersymmetry [10]. However this is not so in the non-Hermitian case. So it is natural to ask whether there exists any symmetry which relates the two Hamiltonians, i.e., the original and the one obtained by the Darboux transformation. The answer to this question is in the affirmative and it will be shown that the two Hamiltonians are related by pseudo supersymmetry [11]. In other words, the Hamiltonians obtained by intertwining are pseudo supersymmetric partners.

Finally we shall examine the problem of applying the Darboux transformation to models with spontaneously broken $\mathcal{PT}$ symmetry. It is known [12] that models with spontaneously broken $\mathcal{PT}$ symmetry exhibit a complex spectrum and all the energy levels appear as complex conjugate pairs. It will be shown that if the Darboux transformation is applied to such a system one gets a potential with complex energy eigenvalues but as singlets.

The paper is organized as follows: in section 2 we briefly present the Darboux construction; in section 3 we construct new $\mathcal{PT}$ symmetric potentials; in section 4 we show that the partner Hamiltonians are connected by pseudo supersymmetry; in section 5 we examine the nature of the spectrum obtained by applying the Darboux transformation to a potential with spontaneously broken $\mathcal{PT}$ symmetry and finally section 6 is devoted to a discussion.

2. Darboux transformation

To make the paper self-contained we start with a brief review of the Darboux transformation [8, 9]. A particle moving in the potential $v(x)$ (real or complex) is characterized by the Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x). \tag{2}$$

(The units used are $\hbar = 2m = 1$ for convenience.)

If the particle is in the mth state, (i.e., m is the quantum number equal to the number of nodes of the mth eigenfunction $\psi_m(x)$ of the starting potential $v(x)$), and the energy scale is adjusted so that the mth energy eigenvalue is exactly zero ($E_m = 0$), then the Schrödinger equation reads

$$H\psi_m = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)\right)\psi_m = 0. \tag{3}$$

Equation (3) has a potential

$$v(x) = \frac{\psi_m''}{\psi_m} \tag{4}$$

which is regular everywhere, so that the Hamiltonian in (3) may be represented as

$$H = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\psi_m''}{\psi_m}\right). \tag{5}$$

Thus if the general solution $\psi = \psi(x)$ of the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + [\epsilon - v(x)]\psi = 0 \tag{6}$$

is known for all values of ϵ, and for a particular value of $\epsilon = E_m$, the particular solution is ψ_m, then the general solution of the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + [E - u(x)]\phi = 0 \tag{7}$$

with

$$\begin{aligned} u(x) &= \psi_m(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{\psi_m(x)}\right) \\ &= 2\left(\frac{\psi_m'}{\psi_m}\right)^2 - \left(\frac{\psi_m''}{\psi_m}\right) \end{aligned} \tag{8}$$

$$E = \epsilon - E_m \tag{9}$$

for $E \neq 0$ is

$$\begin{aligned} \phi_n(x) &= \psi_m(x)\left\{\frac{\psi_n(x)}{\psi_m(x)}\right\}' \\ &= \psi_n'(x) - \left(\frac{\psi_m'(x)}{\psi_m(x)}\right)\psi_n(x). \end{aligned} \tag{10}$$

Choosing different ψ_m, one obtains a series of non-trivial partners $u(x)$ given by (8), which contain all the states of the original potential $v(x)$ except the mth state, i.e. the one corresponding to the eigenstate ψ_m. It is easy to observe that the partners $u(x)$ and $v(x)$ are related by

$$v(x) = W_m(x)^2 - W_m'(x) \tag{11}$$

$$u(x) = W_m(x)^2 + W_m'(x) \tag{12}$$

where

$$W_m(x) = \left(-\frac{\psi_m'}{\psi_m}\right). \tag{13}$$

Though $\psi_m' \neq 0$ at the nodes x_j, $j = 1, 2, 3, \ldots$ of ψ_m, $W_m(x)$ has singularities at these points. However, since the second derivative ψ_m'' also vanishes at the nodes, from (4) $v(x)$ is regular everywhere. In the case of Hermitian SUSY QM, W_m is the superpotential, and $W_m^2(x) \pm W_m'(x)$ are called SUSY-m partner potentials [9]. However, one can construct the partners for $m = 0$ only, as for non-zero m, W_m becomes singular and as a consequence such potentials are not defined over R but over disjoint intervals, the number of intervals depending on the value of m. Thus, for $m = 1$, there are two potentials, each of them defined on a semi-infinite domain, for $m = 2$ there is one potential on a finite domain between nodes x_1 and x_2, and two potentials on the two semi-infinite domains $(-\infty, x_1]$ and $[x_2, +\infty)$, and so on [9]. In the case of non-Hermitian quantum mechanics, $W_m(x)$ is no longer singular (and it is not the *superpotential* anymore). As a consequence the new potentials do not have singularities on the real axis and are defined on $(-\infty, \infty)$. Thus, if one of the partner potentials is exactly solvable, this formalism enables one to construct an infinite number of exactly solvable, non-trivial partners defined on the entire real line $(-\infty, +\infty)$, unlike in the case of Hermitian quantum mechanics.

3. New exactly solvable $\mathcal{PT}$ symmetric potentials

In this section, we shall construct isospectral partners of the complexified Scarf II potential. This potential is given by

$$V(x) = -V_1\,\mathrm{sech}^2\,x - \mathrm{i}V_2\,\mathrm{sech}\,x\,\tanh x \qquad V_1 > 0 \quad V_2 \neq 0 \tag{14}$$

and it has been studied by various authors as it is not only invariant under $\mathcal{PT}$ symmetry, but also $\mathcal{P}$-pseudo Hermitian. This exactly solvable model has certain interesting properties. It has a discrete spectrum that admits both real as well as complex conjugate energies, depending on the relative strengths of its parameters V_1 and V_2. The normalized wavefunctions for this potential are well known, being given by [12]

$$\psi_n(x) = \frac{\Gamma\left(n - 2p + \frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2} - 2p\right)} z^{-p} (z^*)^{-q} P_n^{-2p-\frac{1}{2},-2q-\frac{1}{2}} (\mathrm{i}\sinh x) \tag{15}$$

where $P_n^{\alpha,\beta}$ are the Jacobi polynomials given by [13]

$$P_n^{\alpha,\beta}(\mathrm{i}\sinh x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} F(-n, n+\alpha+\beta+1; \alpha+1; z) \tag{16}$$

and

$$z = \frac{1 - \mathrm{i}\sinh x}{2} \tag{17}$$

$$p = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + V_1 + V_2} = -\frac{1}{4} \pm \frac{t}{2} \tag{18}$$

$$q = -\frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{1}{4} + V_1 - V_2} = -\frac{1}{4} \pm \frac{s}{2} \tag{19}$$

t and s are defined with only the positive sign in the discriminant in p and q. The energy eigenvalues are obtained as

$$E_n = -(n - p - q)^2 \qquad n = 0, 1, 2, \ldots < \left(\frac{s+t-1}{2}\right). \tag{20}$$

Since $V_1 > 0$, two cases arise for real V_2, depending on the relative strengths of the real and imaginary parts of the potential:

1. *The case* $|V_2| \leqslant V_1 + \frac{1}{4}$. In this case the potential and the wavefunctions are $\mathcal{PT}$ invariant, p and q are real and one gets a real bound state spectrum. In addition to the potential (1), the wavefunctions $\psi_n(x)$ given in (15) are also $\mathcal{PT}$ invariant. Note that due to normalization requirements only the values with the positive sign are allowed in (18) and (19).
2. *The case* $|V_2| > V_1 + \frac{1}{4}$. $\mathcal{PT}$ symmetry is spontaneously broken, as though the potential is $\mathcal{PT}$ invariant, the wavefunctions are no longer so. Either p or q is complex, and all energies occur as complex conjugate pairs. Real energies are conspicuous by their absence.

For purely imaginary V_2, however, potential (1) is real, possessing only real energies.

However, when V_2 has both real and imaginary parts, potential (1) loses its $\mathcal{PT}$ invariance, and so will not be considered in this study.

We now consider the first case when $\mathcal{PT}$ symmetry is unbroken. Now, using the explicit solution for the normalized wavefunction (15), we obtain

$$\begin{aligned} W_m(x) &= -\frac{\psi_m'(x)}{\psi_m(x)} \\ &= (p+q)\tanh x - \mathrm{i}(p-q)\,\mathrm{sech}\, x + \frac{mb}{c}\frac{F(-m+1, b+1; c+1; z)}{F(-m, b; c; z)} \end{aligned} \tag{21}$$

where b and c stand for

$$b = -2p - 2q + 2 \tag{22}$$

$$c = -2p + \tfrac{1}{2}. \tag{23}$$

The exactly solvable potential $U^{(m)}(x)$, which is isospectral to the Scarf II potential (except for the mth state), is obtained from the formula

$$U^{(m)}(x) = W_m^2 + W_m' - \beta_m. \tag{24}$$

In writing the last expression we have made use of the fact that if $v(x)$ and $u(x)$ are isospectral, so are $V(x)$ and $U^m(x)$, given by

$$V(x) = \{v(x) - \beta_m\} \tag{25}$$

$$U^m(x) = \{u(x) - \beta_m\}. \tag{26}$$

For the Scarf II potential, β_m is calculated to be

$$\beta_m = (p+q)^2 - 2m(p+q) + m^2. \tag{27}$$

Thus this approach yields new interesting potentials, with eigenfunctions for this particular case being given by (see equation (10)

$$\phi_n(x) = \left(\frac{P_m P_n' - P_m' P_n}{P_m}\right)\psi_0(x). \tag{28}$$

In the above P_n stands for $P_n^{\alpha,\beta}(\mathrm{i}\sinh x)$ and P_n' denotes differentiation of $P_n^{\alpha,\beta}(\mathrm{i}\sinh x)$ with respect to x. Since $P_n^{\alpha,\beta}$ is well defined on the entire real line, so also is $\phi_n(x)$.

Let us analyse three low lying cases $m = 0, 1, 2$.

For $m = 0$

$$U^{(0)}(x) = -\{2(p^2+q^2) - (p+q)\}\,\mathrm{sech}^2 x - \mathrm{i}(p-q)[2(p+q)-1]\,\mathrm{sech}\,x\tanh x \tag{29}$$

with eigenenergies

$$E_n = -(n+1-p-q)^2 \qquad n = 0, 1, 2, 3, \ldots \tag{30}$$

and the ground state

$$\phi_0 = N_0\left(\frac{1-\mathrm{i}\sinh x}{2}\right)^{-(p-\frac{1}{2})}\left(\frac{1+\mathrm{i}\sinh x}{2}\right)^{-(q-\frac{1}{2})}. \tag{31}$$

Thus for $m = 0$, the partners belong to the family of the so-called *satellite* potentials. Equation (29) is also a Scarf II potential, with a different set of parameters, and shares all the energies of (1) except for the ground state of $V(x)$, which is missing in (29). So, this is analogous to the Hermitian case.

$m = 1$ gives the first of the non-trivial potentials.

$$\begin{aligned} U^{(1)}(x) = {} & -\{2(p^2+q^2) - (p+q)\}\,\mathrm{sech}^2 x - \mathrm{i}(p-q)[2(p+q)-1]\,\mathrm{sech}\,x\tanh x \\ & + 2\left(\frac{f_1'}{f_1}\right)^2 - \frac{2(p-q)}{f_1} - 2 \end{aligned} \tag{32}$$

with

$$\begin{aligned} f_1(x) &= F\left(-1, -p-q-\lambda; 2p+\frac{3}{2}; z\right) \\ &= \left\{-(p-q) + \frac{\mathrm{i}}{2}(-2p-2q+1)\sinh x\right\}. \end{aligned} \tag{33}$$

The ground state of the partner (32) is obtained from (10) as

$$\phi_0 = \frac{2\mathrm{i}\left(\frac{1}{2} - p - q\right)}{(-p-q) + \mathrm{i}\left(\frac{1}{2} - p - q\right)\sinh x} z^{-(p-\frac{1}{2})}(z^*)^{-(q-\frac{1}{2})} \tag{34}$$

with energy

$$E_0 = -(-p-q)^2. \tag{35}$$

It can be shown that the state corresponding to ψ_1 is excluded from the spectrum as it turns out to be non-normalizable. All other states share identical energies with the original potential (1). The excited states are obtained from (10) as

$$\phi_n = \left(\frac{P_1 P'_{n+2} - P'_1 P_{n+2}}{P_1}\right)\psi_0 \tag{36}$$

with energies

$$E_n = -(n+2-p-q)^2 \qquad n = 0, 1, 2, 3, \ldots. \tag{37}$$

As $P_n^{\alpha,\beta}(\mathrm{i}\sinh x)$ has no zeroes on the real line, so the eigenfunctions are well defined. It is easily observed from (36) that ϕ_n are normalizable. Moreover, the potential $U^{(1)}(x)$ so constructed has no singularity on the real line, and hence is defined on the entire domain $(-\infty, +\infty)$. Also, for real values of the parameters p and q (corresponding to real energies) the new potential, too, is invariant under $\mathcal{PT}$ transformation.

In an analogous way, the isospectral partner for $m = 2$ is found to be

$$\begin{aligned} U^{(2)}(x) = &-\{2(p^2+q^2) - (p+q)\}\,\mathrm{sech}^2 x - \mathrm{i}(p-q)\,[2(p+q)-1]\;\mathrm{sech}\,x\tanh x \\ &+2\left(\frac{f'_2}{f_2}\right)^2 + \frac{\sigma - 6(p-q)\mathrm{i}\sinh x}{f_2} - 8 \end{aligned} \tag{38}$$

with

$$f_2(x) = F\left(-2, -p-q-\lambda; 2p+\frac{3}{2}; z\right) \tag{39}$$

$$\begin{aligned} \sigma = \frac{-2p-2q+2}{\left(-2p+\frac{1}{2}\right)\left(-2p+\frac{3}{2}\right)} &\left\{2(p-q)^2 - \frac{\left(-2q+\frac{3}{2}\right)\left(-2p+\frac{3}{2}\right)}{(-2p-2q+2)}\right. \\ &\left. + \frac{1}{2}(3+2p+2q)(3-2p-2q)\right\}. \end{aligned} \tag{40}$$

For a visual representation as well as for comparison, we have plotted the real and imaginary parts of the potentials $V, U^{(i)}$ $(i = 0, 1, 2)$. In figure 1, we have plotted the real parts of $V, U^{(i)}$ $(i = 0, 1, 2)$ for the parameter values $V_1 = 24$, $V_2 = 18$ while in figure 2, we have plotted the imaginary parts of the same potentials for the same values of V_1 and V_2.

Since the Scarf II potential is always $\mathcal{PT}$ symmetric, so are its partners $U^{(m)}(x)$, for real values of the parameters p and q $(V_1 + 1/4 \geqslant |V_2|)$, as the functions $f_m(x)$ remain invariant under $\mathcal{PT}$. Moreover, all the new potentials so constructed are defined over the entire domain $(-\infty, +\infty)$, admit real bound state spectrum, and possess all the energies of the original potential except for the mth state, if one starts with the mth order eigenfunction. Though $m = 0$ gives the usual shape-invariant form, highly non-trivial, non-shape-invariant potentials are obtained for non-zero m.

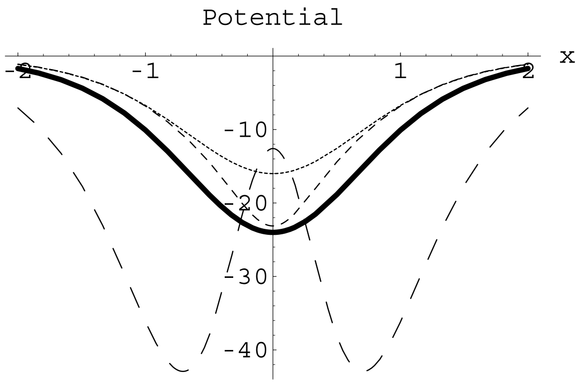

Figure 1. Graph of real parts of V (solid), $U^{(0)}$ (dotted), $U^{(1)}$ (small dash), $U^{(2)}$ (large dash).

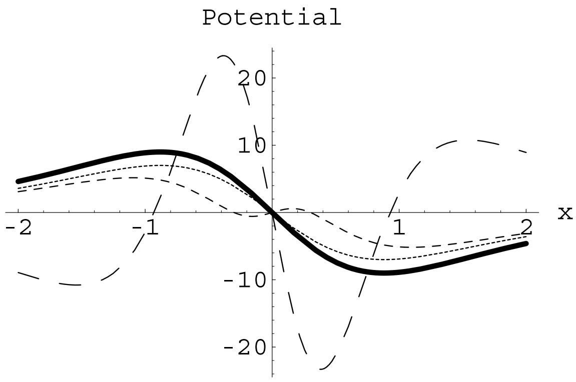

Figure 2. Graph of imaginary parts of V (solid), $U^{(0)}$ (dotted), $U^{(1)}$ (small dash), $U^{(2)}$ (large dash).

4. Pseudo supersymmetry and intertwining

It is well known that in the case of Hermitian quantum mechanics, the Darboux transformation and supersymmetric quantum mechanics are equivalent. Although this is not so in the case of non-Hermitian quantum mechanics, the Darboux transformation can still be implemented in terms of intertwining operators. To see this we consider the intertwining operators A and B:

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W_m \tag{41}$$

$$B = -\frac{\mathrm{d}}{\mathrm{d}x} + W_m \tag{42}$$

where W_m is defined by (13), then the partner Hamiltonians

$$H_{\pm} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W_m^2 \pm W_m' \tag{43}$$

can be written as $H_- = BA$ and $H_+ = AB$, where

$$H_- = \left(-\frac{d^2}{dx^2} + v(x)\right) \tag{44}$$

$$H_+ = \left(-\frac{d^2}{dx^2} + u(x)\right). \tag{45}$$

Evidently, if ψ_n is an eigenfunction of H_- with energy eigenvalue E_n^-, then $\phi_n = A\psi_n$ is also an eigenfunction of H_+ with the same eigenvalue E_n^-, except for $n = m$, since in this case $A\psi_m = 0$.

$$H_+ A\psi_n = (AB)A\psi_n = A(H_-\psi_n) = E_n^-(A\psi_n). \tag{46}$$

For Hermitian Hamiltonians, A and B are mutually adjoint operators ($B = A^\dagger$), giving the well-known results of supersymmetry, namely, $AH_- = H_+A$ or $H_-A^\dagger = A^\dagger H_+$.

To extend the idea of supersymmetry to include non-Hermitian Hamiltonians, we assume the existence of a linear, invertible, Hermitian operator η, such that [11]

$$B = A^\# = \eta^{-1}A^\dagger\eta. \tag{47}$$

This allows one to rewrite the partner Hamiltonians as

$$H_+ = B^\# B \qquad H_- = BB^\# \tag{48}$$

so that

$$BH_+ = H_-B \qquad H_+B^\# = B^\# H_-. \tag{49}$$

From (49) it is clear that B maps eigenfunctions of H_+ to those of H_- and $A(= B^\#)$ does the converse. Thus the mutually adjoint operators A and $A^\dagger$ of conventional supersymmetric quantum mechanics are replaced by their pseudo supersymmetric counterparts A and B when the potential is non-Hermitian. However, we would like to point out that the choice of η is not unique. To determine a form of η, let us note that a simple representation is given by [11]:

$$\eta = \mathcal{P} \qquad \mathcal{P}f(x) = f(-x). \tag{50}$$

It follows that for real potentials, (1) leads to $B = A^\dagger$, thus reproducing the conventional result of supersymmetry. We note that the above results are quite general since they do not depend on a specific W_m. Clearly the operator η can be found in the same way for higher values of m. We thus conclude that in the complex case the Hamiltonians $H_\pm$ are pseudo supersymmetric partners of each other. Finally to cast the above results in a formal pseudo supersymmetric form, let us define the pseudo supercharges Q and $Q^\#$ in the following way:

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \qquad Q^\# = \eta^{-1}Q^\dagger\eta = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}. \tag{51}$$

Thus the pseudo supercharges Q and $Q^\#$ are nilpotent and they satisfy the following closed algebra:

$$H = \{Q, Q^\#\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \qquad [Q, H] = [Q^\#, H] = 0. \tag{52}$$

We thus conclude that in the non-Hermitian case the Hamiltonians obtained by intertwining are pseudo supersymmetric partners.

5. A model with spontaneously broken $\mathcal{PT}$ symmetry

As mentioned earlier, there are two cases for the $\mathcal{PT}$ symmetric Scarf II potential, depending on the relative strengths of V_1 and V_2, namely

(i) $|V_2| \leqslant V_1 + \frac{1}{4}$:
In this case, where the spectrum is real and discrete, each state is a singlet. This has already been discussed in section 3.

(ii) $|V_2| > V_1 + \frac{1}{4}$:
In this case, $\mathcal{PT}$ symmetry is spontaneously broken. We can choose p to be real, taking a single value with only the positive sign in (18) while q can take either of the following values:

$$q^{\pm} = -\frac{1}{4} \pm \mathrm{i}\frac{s}{2} \tag{53}$$

where

$$s = \sqrt{V_2 - V_1 - \tfrac{1}{4}} \tag{54}$$

giving rise to complex conjugate pairs of energies

$$E_n^{\pm} = -\mu_n^2 \pm \mathrm{i}\,\mu_n s \qquad n = 0, 1, 2, \ldots \qquad \mu_n = n - p + \tfrac{1}{4}. \tag{55}$$

This case makes quite an interesting study and we shall investigate it further. Though the original potential (14) is still $\mathcal{PT}$ invariant, the partners are no longer so. We note that while the original potential $V(x)$ does not depend explicitly on the parameters p and q, the partner potential is explicitly dependent on these parameters. As a consequence there are two partner potentials corresponding to $V(x)$ (this is due to the fact that in this case both the values of q are allowed). If the Darboux transformation is carried out by the mth eigenstate ψ_m^- (ψ_m^+) of the original potential, then straightforward calculations show that the corresponding state will be missing in the partner. Furthermore, the entire positive (negative) sector $E_n^+ = -\big(\mu_n^2 - \frac{s^2}{4}\big) + \mathrm{i}\,\mu_n s$ $\big(E_n^- = -\big(\mu_n^2 - \frac{s^2}{4}\big) - \mathrm{i}\,\mu_n s\big)$ will be absent in the partner. Thus the spectrum of the partner potential is quite different from that of the Scarf II potential, as the former has only singlet complex energies. So, if one starts with the ground-state eigenfunction ψ_0^-, then the partner potential (29) is of the form

$$\begin{aligned} U_-^0(x) = &-\left\{\left(2p^2 - p - \frac{s^2}{2} + \frac{3}{8}\right) + \mathrm{i}s\right\}\operatorname{sech}^2 x \\ &- \left\{\mathrm{i}\left(2p^2 - p + \frac{s^2}{2} - \frac{3}{8}\right) + s\right\}\operatorname{sech} x \tanh x \end{aligned} \tag{56}$$

while the wavefunctions and the corresponding energy values are given by

$$\begin{aligned} \phi_n^-(x) &= \frac{ab}{c} z^{-p} (z^*)^{-\frac{1}{4}+\frac{\mathrm{i}s}{2}} F(a+1, b+1; c+1; z) \\ E_n^- &= -\mu_{n+1}^2 - \mathrm{i}\,\mu_{n+1} s \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{57}$$

where

$$a = -(n+1) \qquad b = \big(n + \tfrac{3}{2} - 2p\big) + \mathrm{i}\,s \qquad c = -2p + \tfrac{1}{2}. \tag{58}$$

6. Discussion

In this paper, we have explored the idea of applying the Darboux transformation to non-Hermitian quantum mechanical systems. In particular we have obtained a series of new exactly solvable $\mathcal{PT}$ symmetric potentials with real bound state spectra. The symmetry aspect of the potentials has also been investigated and it has been shown that they are pseudo supersymmetric. On the other hand when the Darboux transformation is applied to a system with spontaneously broken $\mathcal{PT}$ symmetry the resulting potential admits a single tower of energy. We would like to point out that this is not a characteristic feature only of models with spontaneously broken $\mathcal{PT}$ symmetry but in general the same result would be obtained whenever the original model posseses a doublet of states [14].

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